Further Mathematics 2017

Core: Recursion and Financial Modelling

Recurrence Relations, Interest and Depreciation

Key knowledge

- the concept of a first-order linear recurrence relation and its use in generating the terms in a sequence
- the use of first-order linear recurrence relations to model flat rate and unit cost and reducing balance depreciation of an asset over time, including the rule for the future value of the asset after \( n \) depreciation periods
- the concepts of financial mathematics including simple and compound interest, and depreciation

Key skills

- use a given first-order linear recurrence relation to generate the terms of a sequence
- model and analyse growth and decay in financial contexts using a first-order linear recurrence relation of the form \( u_0 = a, u_{n+1} = bu_n + c \)
- demonstrate the use of a recurrence relation to determine the depreciating value of an asset or the future value of an investment or a loan after \( n \) time periods, including from first principles for \( n \leq 5 \)
- use a rule for the future value of a compound interest investment or loan, or a depreciating asset, to solve practical problems

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More resources available at

http://drweiser.weebly.com
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1. First order linear Recursion relationships

A list of numbers, written down in succession, is called a sequence. Each of the numbers is called a term. We write the terms of a sequence as a list, separated by commas. If a sequence continues indefinitely, or if there are too many terms to write them all, we use an ellipsis, ‘…’, at the end of a few terms of the sequence like this:

12, 22, 5, 6, 16, 43, …

The terms in this sequence of numbers could be the ages of the people boarding a plane. The age of these people is random so this sequence of numbers is called a random sequence. There is no pattern or rule that allows the next number in the sequence to be predicted. Some sequences of numbers do display a pattern. For example, this sequence:

1, 3, 5, 7, 9, …

has a definite pattern and so this sequence is said to be rule-based. This sequence of numbers has a starting value. We add 2 to this number to generate the term 3. Then, add 2 again to generate the term 5, and so on. The rule is ‘add 2 to each term’.

![Sequence example](image)

**Example 2** Generating a sequence of terms

Write down the first five terms of the sequence with a starting value of 10 and the rule ‘double the number and then subtract 3’.

**Solution**

1. Write down the starting value.  
2. Apply the rule (double, then subtract 3) to generate the next term. 
3. Calculate three more terms. 
4. Write your answer.

<table>
<thead>
<tr>
<th>Step</th>
<th>Action</th>
<th>Calculation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Start</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>Multiply by 2</td>
<td>2 * 10 - 3</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>Multiply by 2</td>
<td>2 * 17 - 3</td>
<td>31</td>
</tr>
<tr>
<td>4</td>
<td>Multiply by 2</td>
<td>2 * 31 - 3</td>
<td>59</td>
</tr>
</tbody>
</table>

The sequence is 10, 17, 31, 59, …

Using a calculator to generate a sequence of numbers from a rule

All of the calculations to generate sequences from a rule are repetitive. The same calculations are performed over and over again – this is called recursion. A calculator can perform recursive calculations very easily, because it automatically stores the answer to the last calculation it performed, as well as the method of calculation.

**Example 3** Generating a sequence of numbers with a calculator

Use a calculator to generate the first five terms of the sequence with a starting value of 5 and the rule ‘double and then subtract 3’.

**Solution**

Start with a blank calculator page

Press

- [ON] Home
- [1] New document
- [1] add calculator
- Type 5 [5]
- press enter

Starting term

![Calculator example](image)
Generating the terms of a first-order recurrence relations

A **first-order recurrence relation** relates a term in a **sequence** to the previous term in the same sequence. To generate the terms in the sequence, only the **initial term** is required. A recurrence relation is a mathematical rule that we can use to generate a sequence. It has two parts:

1. a **starting point**: the value of one of the terms in the sequence
2. a **rule** that can be used to generate successive terms in the sequence.

For example, in words, a recursion rule that can be used to generate the sequence: 10, 15, 20, … can be written as follows:

1. Start with 10.
2. To obtain the next term, add 5 to the current term and repeat the process.

A more compact way of communicating this information is to translate this rule into symbolic form. We do this by defining a subscripted variable. Here we will use the variable $V_n$, but the V can be replaced by any letter of the alphabet.

Let $V_n$ be the term in the sequence **after n iterations**.

Using this definition, we now proceed to translate our rule written in words into a mathematical rule.

<table>
<thead>
<tr>
<th>Starting value $(n=0)$</th>
<th>Rule for generating the next term</th>
<th>Recurrence relation (two parts: starting value plus rule)</th>
</tr>
</thead>
</table>
| $V_0=10$               | $V_{n+1}=V_n+5$                  | $V_0=10$  
Next term =current term +5  
Starting value  
rule |

**Note:** Because of the way we defined $V_n$, the starting value of $n$ is 0. At the start there have been no applications of the rule. This is the most appropriate starting point for financial modelling.

*Each time we apply the rule it is called an iteration.*
Example 5 Using a calculator to generate sequences from recurrence relations

A sequence is generated by the recurrence relation \( V_0 = 300, V_{n+1} = 0.5V_n - 9. \)

Use your calculator to generate this sequence and determine how many terms of the sequence are positive.

**Solution**

Start with a blank calculator page

Press

- Type 300 [enter]
- Press enter

Next

- Type \( \times \) 0.5 - 9 [enter]
- Press enter

Continue to press [enter] until the first negative term appears and write your answer:
- The first 5 terms of the sequence are positive.

The importance of the Starting Term

In the example above, if the same rule is used with a different starting point, it will generate different sets of numbers.

Example 5 \( V_0 = 300, V_{n+1} = 0.5V_n - 9 \) The first five terms were: 300, 141, 61.5, 21.75, 1.875

If \( V_0 = 250 \) then, the first 5 terms would be: 250, 116, 49, 15.5, -1.25

Finding other Terms in a recurrence relation

We can also use recurrence relations to find previous terms, but we need two pieces of information

1. The rule, in terms of \( V_{n+1} \) and \( V_n \)
2. The term number and its value. i.e. \( n = 2 \) and \( V_2 = 10 \) (note if \( n = 0, 1, 2, \ldots \) then \( n = 2 \) is the 3rd term)

First-order linear recurrence relations with a common difference

The common difference, \( d \), is the value between consecutive terms in the sequence:

Look at the sequence 3, 7, 11, 15, 19, ... .

\[ d = u_2 - u_1 = u_3 - u_2 = u_4 - u_3 = \ldots \]
\[ d = 7 - 3 = 11 - 7 = 15 - 11 = +4 \]

The common difference is +4.

This sequence may be defined by the first-order linear recurrence relation:

\[ u_{n+1} - u_n = 4 \quad u_0 = 3 \]

Rewriting this

\[ u_{n+1} = u_n + 4 \quad u_0 = 3 \]

A sequence with a common difference of \( d \) may be defined by a first-order linear recurrence relation of the form:

\[ u_{n+1} = u_n + d \quad (or \ u_{n+1} - u_n = d) \]

where \( d \) is the common difference and for

- \( d > 0 \) it is an increasing sequence
- \( d < 0 \) it is a decreasing sequence.
Example 6

Express each of the following sequences as first-order recurrence relations.

a) 7, 12, 17, 22, 27, ...

Using CAS Calculator

To check if there is a common difference

Use a list and spreadsheet page
- Enter the values in the first column

Modelling linear growth and decay

Linear growth and decay is commonly found around the world. They occur when a quantity increases or decreases by the same amount at regular intervals. Everyday examples include the paying of simple interest or the depreciation of the value of a new car by a constant amount each year.

An example of linear growth is the investment of money, such as putting it in a savings account where the sum increases over time.

An example of linear decay is the money owned to repay a loan, the sum of money owned will decrease over time.

A recurrence model for linear growth and decay

The recurrence relations

\[ P_0 = 20, \quad P_{n+1} = P_n + 2 \quad \quad Q_0 = 20, \quad Q_{n+1} = Q_n - 2 \]

both have rules that generate sequences with linear patterns, as can be seen from the table below. The first generates a sequence whose successive terms have a linear pattern of growth, and the second a linear pattern of decay.
Generally, if $d$ is a constant, a recurrence relation rule of the form:

\[ V_{n+1} = V_n + d \]

can be used to model **linear growth**.

\[ V_{n+1} = V_n - d \]

can be used to model **linear decay**.

### Exercise 5.1: Generating a sequence recursively

1. Use the following starting values and rules to generate the first five terms of the following sequences recursively **by hand**.

   a) Starting value: 2  rule: add 6  
   b) Starting value: 5  rule: subtract 3  
   c) Starting value: 1  rule: multiply by 4  
   d) Starting value: 10  rule: divide by 2  
   e) Starting value: 6  rule: multiply by 2 add 2  
   f) Starting value: 12  rule: multiply by 0.5 add 3

2. Use the following starting values and rules to generate the first five terms of the following sequences recursively **using a CAS calculator**.

   a) Starting value: 4  rule: add 2  
   b) Starting value: 24  rule: subtract 4  
   c) Starting value: 2  rule: multiply by 3  
   d) Starting value: 50  rule: divide by 5  
   e) Starting value: 5  rule: multiply by 2 add 3  
   f) Starting value: 18  rule: multiply by 0.8 add 2
2. Simple Interest (Textbook Chapter 6.2)

When people wish to purchase an item they cannot afford but know they can pay for the item in a certain time they have a few options: credit card (at high interest rates), lay-by (where they gradually pay off the item but cannot use it until the final payment is made), hire purchase (making regular repayments with use of the item) or a loan (from a bank or credit union).

The last two options frequently attached simple interest. This is the cost of using the banks money. Investments also utilise simple interest. If you deposit money in an account the bank will use your money and will pay interest into your account (for using your money).

Simple interest can be represented by a first-order linear recurrence relation.

\[
V_{n+1} = V_n + d, \quad d = \frac{V_0 \times r}{100},
\]

where \(V_n\) represents the value of the investment after \(n\) time periods, \(d\) is the amount of interest earned per period, \(V_0\) is the initial (or starting) amount and \(r\) is the interest rate.

You can also calculate the total amount of a simple interest loan or investment by using:

\[
\text{Total amount of loan or investment} = \text{initial amount or principal} + \text{interest}
\]

\[
V_n = V_0 + I
\]

where:

\[
I = \frac{V_0 r n}{100} \quad I = \text{simple interest charged or earned (\$)}
\]

\[
V_0 = \text{principal (money invested or loaned) (\$)}
\]

\[
r = \text{rate of interest per period (\% per period)}
\]

\[
n = \text{the number of periods (years, months, days) over which the agreement operates}
\]

In the case of simple interest, the total value of investment increases by the same amount per period. Therefore, if the values of the investment at the end of each time period are plotted, a straight-line graph is formed.
Worked Example 1

$325 is invested in a simple interest account for 5 years at 3% p.a.

a) Set up a recurrence relation to find the value of the investment after \( n \) years.

\[
\begin{array}{|c|c|c|}
\hline
n & V_n ($) & V_{n+1} ($) \\
\hline
0 & V_0 = 325 & \blank \\
1 & \blank & \blank \\
2 & \blank & \blank \\
3 & \blank & \blank \\
4 & \blank & \blank \\
\hline
\end{array}
\]

b) Use the recurrence relation from part (a) to find the value of the investment at the end of each of the first 5 years.

Worked Example 1(b) on CAS calculator

Start with a blank calculator page
Press
• Type 325
• press enter

Next type
\[ \begin{array}{c}
+0.03 \times 325 \\
\times 325
\end{array} \]
Press enter

Note: when you press enter, the CAS converts ANS to the value of the previous answer (in this case 325)

Pressing enter repeatedly applies the rule “+0.03x325” to the last calculated value, in the process generating the amount of the investment at the end of each year as shown.

1\(^{\text{st}}\) year
2\(^{\text{nd}}\) year
3\(^{\text{rd}}\) year
4\(^{\text{th}}\) year
5\(^{\text{th}}\) year

1st year
2nd year
3rd year
4th year
5th year
Worked Example 2

Jan invests $210 with building society in a fixed deposit account that paid 8% p.a. simple interest for 18 months.

a) How much did she receive after the 18 months?

b) Represent the account balance for each of the 18 months graphically.

---

**Worked Example 2(b) on CAS Calculator**

- **Label column A “month”**
  - Enter 0 in cell A1
  - In cell A2 enter: =A1+1
  - Fill down until the 18th month

- **Label column B “total”**
  - Enter $210 in cell B1
  - In the next cell (B2) enter the equation =210+1.4*A2
  - Now fill down this equation to the cells below.

- **Add a data and statistics page**
  - Put the “month” on the x axis
  - and “total” on the y axis
**Finding \( V_0, r \) and \( n \)**

**Worked Example 3**

A bank offers 9% p.a. simple interest on an investment. At the end of 4 years the total interest earned was $215. How much was invested?

---

**Worked Example 3 on CAS calculator**

On a calculator use the nSolve function, Enter the equation \( I = \frac{V_0 \times r \times n}{100} \), and set the values of \( I, r \) and \( n \) using “|”

Press enter to get the answer of $597.22

---

**Transposed simple interest formula**

- **To find the principal:** \( V_0 = \frac{100 \times I}{r \times n} \)
- **To find the interest rate:** \( r = \frac{100 \times I}{V_0 \times n} \)
- **To find the period of the loan or investment:** \( n = \frac{100 \times I}{V_0 \times r} \)

---

**Worked Example 4**

When $720 is invested for 36 months it earns $205.20 simple interest. Find the yearly interest rate.
Worked Example 4 on CAS calculator

On a calculator use the nSolve function,

Enter the equation \( I = \frac{V_o \times r \times n}{100} \),

and set the values of I, \( V_o \) and n

using “|”, \( I=205.20 \), \( V_o=720 \) and \( n=3 \)

Worked Example 5

An amount of $255 was invested at 8.5% p.a. How long will it take, to the nearest year, to earn $86.70 in interest?

Worked Example 5 on CAS calculator

On a calculator use the nSolve function,

Enter the equation \( I = \frac{V_o \times r \times n}{100} \),

and set the values of I, \( V_o \) and r using “|”, \( I=86.70 \), \( V_o=255 \) and \( r=8.5 \)
3. Flat rate depreciation (6.6 in textbook)

Some items such as antiques, jewellery and real estate increase in value (appreciate or increase in capital gain). Computers, mobile phones, cars or machinery decrease in value (depreciate) with time due to wear and tear, advances in technology or lack of demand.

Depreciation is the estimated loss in value of assets. The estimated value of an item at a point in time is called its future value (book value).

When the value becomes zero, the item is written off. At the end of an item’s useful life its future value is called its scrap value.

Future value = cost price – total depreciation to that time
When book value = $0, then the item is said to be written off.
Scrap value is the book value of an item at the end of its useful life.

There are 3 methods in which to calculate depreciation:

1. flat rate depreciation
2. reducing balance depreciation
3. unit cost depreciation

Flat rate (straight line depreciation)

If an item depreciated by the flat rate method, then the value decreases by a fixed amount each time interval. It may be expressed in dollars or as a percentage of the original cost price.

As the depreciation value is the same for each interval, it is an example of straight line decay. This relationship can be expressed in the following recurrence relation:

\[ V_{n+1} = V_n - d \]

where \( V_n \) is the value of the asset after \( n \)-depreciating periods and \( d \) is the depreciation each time period.

The future value can also be calculated after \( n \) periods of depreciation.

\[ V_n = V_0 - nd \]

We can use the above relationship or a depreciation schedule (table) to analyse flat rate depreciation.

Worked Example 13

Fast Word Printing Company bought a new printing press for $15 000 and chose to depreciate it by the flat rate method. The depreciation was 15% of the prime cost each year and its useful life was 5 years.

a) Find the annual depreciation.
b) Set up a recurrence relation to represent the depreciation

c) Draw a depreciation schedule for the useful life of the press and use it to draw a graph of book value against time.

<table>
<thead>
<tr>
<th>Time $n$ (years)</th>
<th>Depreciation $d$ ($)</th>
<th>Future value $V_n$ ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
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<tr>
<td>3</td>
<td></td>
<td></td>
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<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
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</tr>
</tbody>
</table>

![Graph](image)

![Graph](image)

d) Generate the relationship between the book value and time and use it to find the scrap value.
**Worked Example 13(c) and (d) on CAS calculator**

**13(c) On a lists & spreadsheet page**
- Label column A “n” and column B “Vn”
- Enter 0 to 4 in the n column and the starting value 15000 (V0) in cell b1.

![CAS calculator screenshot showing column labels and values](image)

**In cell b2**
- Enter the equation “=b1-2250”
This equation is just Vn+1=Vn-2250 found in part (b)
Note: the 2250 is the annual depreciation found in part (a)

![CAS calculator screenshot showing cell b2](image)

**Press enter, then**
- fill down (menu [3][3]) until n=5
Vn=3750 when n=5. So, this is the scrap value

![CAS calculator screenshot showing fill down](image)

**Add a Data & Statistics page**
- Label the x-axis “n” and the y-axis “Vn”

![Graph showing data points](image)

In this worked example the depreciation schedule gives the scrap value, when n=5 Vn=$3750. This can also be seen in the graph of book value against time, since it is only drawn for the item’s useful life and its end point is the scrap value.

In Australia, businesses need to keep records of depreciation of all their assets on a year- to-year basis, for tax purposes.

What if you want to investigate the rate at which an item has depreciated over many years? A car, computer or mobile phone? If a straight line depreciation model is chosen, then the following example demonstrates its application.
Worked Example 14

Jarrod bought his car 5 years ago for $15 000. Its current market value is $7500. Assuming straight line depreciation, find:

a) the car’s annual depreciation rate

b) the relationship between the future value and time, and use it to find when the car will have a value of $3000.

Worked Example 14 on CAS calculator

On a lists & spreadsheet page
- Label column A “n” and enter 0 in cell a1, 1 in cell a2 etc, or in cell a2 enter “=a1+1 and fill down until n=10.

Label column B “Vₙ” in cell b2
- Enter the equation =15000−n×1500
  This equation is just \( V_{n+1} = V_n - n \times d \), where \( d=1500 \) and \( V_0=15000 \)

Press enter, the CAS needs to know if “n” is column n or a variable, IT IS A VARIABLE

Click OK and the values for Vn will be shown
· Scroll down until it is 3000, and the value of n is 8
4. Compound Interest Tables (Textbook 6.3)

For investments, when interest is added to the initial amount (principal) invested at the end of an interest-bearing period, and then both the principal and interest earn further interest during the next period, which in turn is added to the balance. This process continues for the life of the investment. The interest is said to be compounded.

Both the balance of the account and interest increase at regular intervals.

Example

Consider $1000 invested for 4 years at an interest rate of 12% p.a. with interested compounded annually. What will be the final balance of the account?

<table>
<thead>
<tr>
<th>Time period ((n))</th>
<th>(V_n($))</th>
<th>Interest ($)</th>
<th>(V_{n+1} ($))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
<td>12% of 1000 = 120</td>
<td>1000+120 = 1120</td>
</tr>
<tr>
<td>1</td>
<td>1120</td>
<td>12% of 1120 = 134.40</td>
<td>1120+134.40 = 1254.40</td>
</tr>
<tr>
<td>2</td>
<td>1254.40</td>
<td>12% of 1254.40 = 150.53</td>
<td>1254.40+150.53 = 1404.93</td>
</tr>
<tr>
<td>3</td>
<td>1404.93</td>
<td>12% of 168.59 = 168.59</td>
<td>1404.93+168.59 = 1573.52</td>
</tr>
<tr>
<td>4</td>
<td>1573.52</td>
<td>12% of 1573.52 = 188.82</td>
<td>1573.52+188.82 = 1762.34</td>
</tr>
</tbody>
</table>

So the balance after 5 years is $1762.34.

In the above example the principle is increased by 12% per year. That is at the end of year balance is 112% or 1.12 of the start of year balance.

<table>
<thead>
<tr>
<th>Time period</th>
<th>Balance($)</th>
</tr>
</thead>
</table>
| 1           | 1120 = 1000 \times 1.12  
= 1000 \times (1.12)^1 |
| 2           | 1254.40 = 1120 \times 1.12 
= 1000 \times 1.12 \times 1.12 
= 1000 \times (1.12)^2 |
| 3           | 1404.93 = 1254.40 \times 1.12 
= 1000 \times 1.12 \times 1.12 \times 1.12 
= 1000 \times (1.12)^3 |
| 4           | 1573.52 = 1404.93 \times 1.12 
= 1000 \times 1.12 \times 1.12 \times 1.12 \times 1.12 
= 1000 \times (1.12)^4 |
| 5           | 1762.34 = 1573.52 \times 1.12 
= 1000 \times 1.12 \times 1.12 \times 1.12 \times 1.12 \times 1.12 
= 1000 \times (1.12)^5 |
Worked Example 6

Laura invested $2500 for 5 years at an interest rate of 8% p.a. with interest compounding annually. Complete the table by calculating the values A, B, C, D, E and F.

<table>
<thead>
<tr>
<th>Time period $(n + 1)$</th>
<th>$V_n$ ($)</th>
<th>Interest ($)</th>
<th>$V_{n+1}$ ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2500</td>
<td>A% of 2500 = 200</td>
<td>2700</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>8% of C = 216</td>
<td>D</td>
</tr>
<tr>
<td>3</td>
<td>2916</td>
<td>8% of 2916 = 233.28</td>
<td>3149.28</td>
</tr>
<tr>
<td>4</td>
<td>3149.28</td>
<td>8% of 3149.28 = 251.94</td>
<td>E</td>
</tr>
<tr>
<td>5</td>
<td>F</td>
<td>8% of 3401.22 = 272.10</td>
<td>3673.32</td>
</tr>
</tbody>
</table>

Worked Example 6 on CAS calculator

Enter the labels “n+1”, “$V_n$”, “Interest”, “$V_{n+1}$”

Note: You can’t use + on the CAS so spell it out

Next enter 1 to 5 in column A, and the starting values for $V_n$=2500, Interest=200 and $V_{n+1}$=2700 in cells b1, c1 and d1 respectively.

Then enter formulas shown below into cells b2, c2 and d2

Now fill down the equations of cells b2, c2 and d2, downward for each of columns b, c and d.

The last screen picture shows the completed table.
5. The Compound Interest Formula (Textbook 6.4)

Geometric Growth

Not all sequences have a common difference (increasing/decreasing by adding/subtracting the same difference to find the next term). The sequence may increase/decrease by multiplying the terms by a common ratio.

Consider the geometric sequence 1, 3, 9, 27, 81, ...

The common ratio can be found by dividing the current term by the previous term. So generally:

\[ R = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \cdots \]

And in this example:

\[ R = \frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \cdots \]

Here the common ratio is 3.

The sequence can be defined by the first-order linear recurrence relation:

\[ u_{n+1} = 3u_n \quad \text{where: } u_0 = 1 \]

A sequence with a common ratio of \( R \) may be defined by a first-order linear recurrence relation of the form:

\[ u_{n+1} = Ru_n \]

where \( R \) is the common ratio

- \( R > 1 \) is an increasing sequence
- \( 0 < R < 1 \) is a decreasing sequence
- \( R < 0 \) is a sequence alternating between positive and negative values.

Compound Interest is Non-linear Growth

This is because the rule governing compound interest is an increase by the same amount (Interest in %) at regular intervals. In Compound Interest \( R > 1 \). Consider the following recurrence relations:

From the previous section, we can see that we could write the value of the investment in terms of its previous value and hence, express it as the recurrence relation:

\[ V_{n+1} = V_nR \]

where \( V_{n+1} \) is the amount of the investment 1 time period after \( V_n \), \( R \) is the growth or compounding factor \( \left( 1 + \frac{r}{100} \right) \) and \( r \) is interest rate per period.

This pattern can be written in terms of the initial investment. This is the compound interest formula.

\[ V_n = V_0R^n \quad \text{where} \quad V_n = \text{final or total amount (}$) \]
\[ V_0 = \text{principal (}$) \]
\[ R = \text{growth or compounding factor } \left( 1 + \frac{r}{100} \right) \]
\[ r = \text{interest rate per period} \]
\[ n = \text{number of interest-bearing periods} \]

This formula gives the total amount in an account, not just the interest earned.

To find the total interest compound, \( I \):

\[ I = V_n - V_0 \quad \text{where} \quad V_n = \text{final or total amount (}$) \]
\[ V_0 = \text{principal (}$) \]
Worked Example 7

$5000 is invested for 4 years at 6.5% p.a., interest compound annually.

a) Generate the compound interest formula for this investment.

b) Find the amount in the balance after 4 years and the interest earned over this period.

Worked Example 7 on CAS calculator

On a calculator page
Using the Solve function
Enter the compound interest formula
\[ V_n = V_0 \left(1 + \frac{r}{100}\right)^n \]
and set the values of
\[ V_0=5000, \; r=6.5 \; \text{and} \; n=4 \]
using “|*

Top Tip: You could save this document on your CAS and just change the values

Press enter to get the value of \( V_n \)

To find the interest earned, subtract the principal from the balance. \( V_n - V_0 \)
On the CAS enter

\( V_n - V_0 \)

(note: the CAS will insert ANS before the minus)

Press enter to get the answer

*| tells the CAS the values of variables, think of line entry as:
“solve this” (equation with variables) “when” the variables are...Non-annual compounding
Many accounts can be compounded quarterly (every three months), weekly or daily. In these cases $n$ and $r$, are determined as follows:

$$\text{Number of interest periods, } n = \text{number of years} \times \text{number of interest periods per year}$$

$$\text{Interest rate per period, } r = \frac{\text{nominal interest rate per annum}}{\text{number of interest periods per year}}$$

Nominal interest rate per annum is the annual interest rate advertised by a financial institution.

**Worked Example 8**

If $3200$ is invested for 5 years at 6% p.a., interest compounded quarterly:

a) Find the number of interest bearing periods, $n$

b) find the interest rate per period, $r$

c) find the balance of the account after 5 years

d) graphically represent the balance at the end of each quarter for 5 years. Describe the shape of the graph.

The graph is exponential as the interest is added at the end of each quarter and the following interest is calculated on the new balance.
Worked Example 8(c) and 8(d) on CAS calculator

**Top Tip:** Because we want to create a graph in 8(d) we will do this on a “list & spreadsheet” page

- On a “list & spreadsheet” page
- Label column A “n” and column B “Vn”
- In cell \( a1 \) enter “0” and in \( b1 \) enter the \( V_0 \) value of $3200
- In cell \( a2 \) enter the formula =a1+1, and then fill down (menu 3 3) to cell \( a21 \) (from 8(a) \( n=20 \))

![Image of CAS calculator](image1)

- In cell \( b2 \) enter the formula
  \[
  = 3200 \left( 1 + \frac{1.5}{100} \right)^{a2}
  \]

  Note: \( r=1.5 \) is from part (b) \( r = \frac{6}{4} \)
- Press [enter] to get the value of \( V_1 \)

![Image of CAS calculator](image2)

- Then fill down (menu 3 3)

![Image of CAS calculator](image3)

- Add a “data & statistics” page (menu 7 3)

Label the x-axis “n” (Quarters) and the y-axis “Vn” (Balance)

![Image of graph](image4)
Worked Example 9

Find the principal that will grow to $4000 in 6 years, if interest is added quarterly at 6.5% p.a.

Worked Example 9 on CAS calculator

On a calculator page
Using the Solve function
Enter the compound interest formula

\[ V_n = V_0 \left(1 + \frac{r}{100}\right)^n \]

and set the values of
\( V_0 = $4000, \ r = 1.625 \) and \( n = 24 \) using “|”

Top Tip: You could save this document on your CAS and just change the values

Press enter to get the value of \( V_0 \)
6. Finding the rate or time for Compound Interest (Textbook 6.5)

Sometimes we know how much we can afford to invest, as well as the future amount that we require at the end of the investment. This allows us to determine the interest rate required to ensure we reach our target investment (savings) goal. With this information, we can 'shop' around to find the best financial institution that will provide that interest rate.

We must first find the interest rate per period, \( r \), and convert this to the corresponding nominal rate per annum. This and finding the time or number of periods is difficult.

Your CAS has a finance function called Finance Solver. This can be used for compound interest calculations as shown in the worked examples in this section and in the future.

Notes on the use of the Financial Solver

Example of Finance Solver

Find the amount of interest earned if $3200 is invested for 5 years at 6% p.a. compounded quarterly using the Finance Solver.

On a calculator page

Press:

menu menu
8 Finance
1 Finance Solver

Complete the fields as shown.
N is the number of payments (20).
I(%) is the interest rate p.a. (6).
PV is the amount to be invested (-3200)*.
Pmt is the regular payment ($0).
FV is the future value of the investment (to be determined).
PpY is the number of payments per year (4).
CpY is the number of compounding period per year (4).
Press the tab key [tab] to move between fields.

Press the [tab] to return to the FV field and press [enter].

The investment is worth $4309.94 after 5 years.

The interest earned is $4309.94 – $3200 = $1109.94.

*The principal value (PV) is entered as a negative value, because you give it to the bank. Hence the future value (FV) is a positive value to indicate it is given to you by the bank.
**Worked Example 10**

Find the interest rate per annum (correct to 2 decimal places) that would enable an investment of $3000 to grow to $4000 over 2 years if interest is compounded quarterly.

Complete the fields as shown.

N is the number of payments (8=2 years x 4 quarters).

I(%) is the interest rate p.a. (to be determined – clear cell).

PV is the amount to be invested (−$3000) - you give money to bank

Pmt is the regular payment ($0).

FV is the future value of the investment ($4000).

PpY is the number of payments per year (4).

CpY is the number of compounding period per year (4).

Press the tab key [tab] to move between fields.

Press the [tab] to return to the I% field and press [enter].

An annual Interest rate of 14.65% p.a. is required (correct to 2 decimal places).

**Finding time in compound interest**

To find n, the number of interest-bearing periods – the time period of an investment, we will use the Financial solver on the CAS.

Often, the value obtained for n, the number will be a decimal, indicating the investment time is between two integers.

The smaller integer doesn’t allow enough time for the investment to have the required balance and the larger integer represents more than the required time.

**Worked Example 11**

How long will it take $2000 to amount to $3500 at 8% p.a. with interest compounded annually?

Complete the fields as shown.

N is the number of payments (to be determined – clear cell)

I(%) is the interest rate p.a. (8%).

PV is the amount to be invested (−$2000) - money given away

Pmt is the regular payment ($0).

FV is the future value of the investment ($3500).

PpY is the number of payments per year (1).

CpY is the number of compounding period per year (1).

Press the tab key [tab] to move between fields.
Press the [tab] to return to the N field and press [enter].

As the Interest is compounded annually, so \( n \) represents years. Round \( n \) up to the next whole year.

Write your answer in words

“\( \)It will take 8 years for $2000 to increase to $3500.\( \)"

As discussed above, if we leave \( n=7.27 \) years (or worse round it down to \( n=7 \) years) it won’t be long enough time for the investment to have reached the $3500 balance required. So, we often to round-up to the nearest whole number (\( n=8 \)) because after 8 years sufficient interest periods (iterations) will have occurred to surpass the $3500 balance required.

Worked Example 12

Calculate the number of interest –bearing periods, \( n \), required and hence the time it will take $3600 to amount to $5100 at a rate of 7 % p.a., with interest compounding quarterly.

Complete the fields as shown.
N is the number of payments (\textit{to be determined} - clear cell).
I(\%) is the interest rate p.a. (7%).
PV is the amount to be invested (–$3600).
Pmt is the regular payment ($0).
FV is the future value of the investment ($5100).
PpY is the number of payments per year (4).
CpY is the number of compounding period per year (4).
Press the tab key [tab] to move between fields.

Press the [tab] to return to the N field and press [enter].

As the Interest is compounded quarterly, so \( n \) represents quarters. Round \( n \) up to the next whole quarter. So, \( n=21 \) quarters

Write your answer in words

“\( \)It will take 21 quarters or 5 ¼ years for $3600 to increase to $5100.\( \)"
7. Reducing balance depreciation (Textbook 6.7)

If an item depreciates by the reducing balance depreciation method, its value reduces by a fixed value each time period. The rate is a percentage of the previous value of the item.

Reducing balance depreciation can be known as diminishing value depreciation.

Reducing balance depreciation can be expressed by the recurrence relation:

\[ V_{n+1} = RV_n \]

where \( V_n \) is the value of the asset after \( n \) depreciating periods and \( R = 1 - \frac{r}{100} \), where \( r \) is the depreciation rate.

**Worked Example 15**

Suppose the new $15 000 printing press considered in Worked example 13 was depreciated by the reducing balance method at a rate of 20% p.a. of the previous value.

a) Generate a depreciation schedule using a recurrence relation for the first 5 years of work for the press.

<table>
<thead>
<tr>
<th>Time ( n ) (years)</th>
<th>( V_{n+1} = RV_n )</th>
<th>Future value ( V_n ) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( V_0 = 15000 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( V_1 = )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( V_2 = )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( V_3 = )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( V_4 = )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( V_5 = )</td>
<td></td>
</tr>
</tbody>
</table>

b) What is the future value after 5 years?

c) Draw a graph of future value against time.
Worked Example 15 on CAS calculator

On a lists & spreadsheet page

- Label column A “n” and column B “V_n”
- Enter 0 to 5 in the n column and the starting value 15000 (V_0) in cell b1.

![Spreadsheet image showing data entry]

In cell b2

- Enter the equation “=0.8×b1”

Note: This equation is just \( V_{n+1} = R \times V_n \)
where \( R = 0.8, \quad R = 1 - \frac{r}{100}, \quad \text{and} \quad r = 20\% \text{ p.a.} \)

Press enter, then

- fill down \((b33)\) until n=5

V_n=$4915.20 when n=5. So, this is the value of the press after 5 years

![Spreadsheet image showing data after fill down]

Add a Data & Statistics page

- Label the x-axis “n” and the y-axis “V_n”

![Graph showing data]

The Australian Tax Office (ATO) allows depreciation of an asset as a tax deduction, meaning that the depreciation reduces an individuals or businesses amount of tax to be paid. If using the reducing balance method, less tax will be paid at the beginning of the asset’s life compared to the end of the asset’s life, whereas a flat rate depreciation will have the same amount deducted for the asset’s lifetime.
A comparison between the two depreciation methods.

Worked Example 16

A transport business has bought a new bus for $60 000. The business has the choice of depreciating the bus by a flat rate of 20% of the cost price each year or by 30% of the previous value each year.

a) Generate depreciation schedules using both methods for a life of 5 years.

<table>
<thead>
<tr>
<th>Time ( n ) (years)</th>
<th>Depreciation ( d ) ($)</th>
<th>Future value ( V_n ) ($)</th>
<th>Time ( n ) (years)</th>
<th>( V_{n+1} = RV_n )</th>
<th>Future value ( V_n ) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
<td>( V_0 = )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
<td>( V_1 = )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>2</td>
<td>2</td>
<td>( V_2 = )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>3</td>
<td>3</td>
<td>( V_3 = )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>4</td>
<td>4</td>
<td>( V_4 = )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>5</td>
<td>5</td>
<td>( V_5 = )</td>
<td></td>
</tr>
</tbody>
</table>

b) Draw graphs of future value against time for both methods on the same set of axes.

c) After how many years does the reducing balance future value become greater than the flat rate future value?
Reducing balance depreciation formula

The reducing balance depreciation formula is:

\[ V_n = V_0 R^n \]

- \( V_n \) = book value after time, \( n \)
- \( R \) = rate of depreciation \( \left( = 1 - \frac{r}{100} \right) \)
- \( V_0 \) = cost price
- \( n \) = time since purchase

Worked Example 17

The printing press from Worked example 13 was depreciated by the reducing balance method at 20% p.a. What will be the future value and total depreciation of the press after 5 years if it cost $15 000 new.
Effective life

We may know the scrap value of an item and we want to determine how long before the item reaches this value, i.e. it’s useful or effective life.

In this case, we use the reducing balance formula.

Worked Example 18

A photocopier purchased for $8000 depreciates by 25% p.a. by the reducing balance method. If the photocopier has a scrap value of $1200, how long will it be before this value is reached?

Worked Example 18 on CAS calculator

On a calculator page
Using the Solve function
Enter the reducing balance depreciation formula

\[ V_n = V_0 R^n \]

\[ R = \left(1 - \frac{r}{100}\right) \]

and set the values of \( V_n = 1200, \ V_0 = 8000 \) and \( R = 0.75 \)

using \| symbol on the CAS

Top Tip: You could save this document on your CAS and just change the values

Press enter

The answers is \( n = 6.5945 \)

As the depreciation is calculated once a year, we need to round this up to \( n = 7 \) years!

Answer: It will take 7 years for the photocopier to reach its scrap value
8. Unit cost depreciation (Textbook 6.8)

The **unit cost method** is based upon the maximum output (units) of the item. For example the useful life of a truck could be expressed in terms of the distance travelled rather than number of years. The actual depreciation per year would be a measure of the number of kilometres travelled.

**Unit cost depreciation recurrence relation**

The future value over time using unit cost depreciation can be expressed by the recurrence relation:

\[
V_{n+1} = V_n - d
\]

where \( V_n \) is the value of the asset after \( n \) outputs and \( d \) is the depreciation per output.

**Worked Example 19**

A motorbike purchased for $12000 depreciates at a rate of $14 per 100 km driven.

a) Set up a recurrence relation to represent the depreciation.

b) Use the recurrence relation to generate a depreciation schedule for the future value of the bike after it has been driven for 100 km, 200 km, 300 km, 400 km and 500 km.

<table>
<thead>
<tr>
<th>Distance driven (km)</th>
<th>Outputs ( n )</th>
<th>Future value ( V_n ) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td></td>
<td></td>
</tr>
<tr>
<td>400</td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Worked Example 19(b) on CAS calculator

#### On a lists & spreadsheet page

- Label column A “n” and column B “V<sub>n</sub>”
- Enter 0 to 5 in the n column and the starting value 15000 (<i>V<sub>0</sub></i>) in cell b1.

#### In cell b2
- Enter the equation “=c1−14”

**Note:** This equation is just <i>V<sub>n+1</sub></i> = <i>V<sub>n</sub></i> − <i>d</i>

Where <i>V<sub>n</sub></i> is the value of the asset after <i>n</i> outputs and <i>d</i> is the depreciation per output.

Press enter, then
- fill down ([menu]33) until n=5

---

**Worked Example 20**

A taxi is bought for $31 000 and it depreciated by 28.4 cents per kilometre driven. In one year the car is driven 15 614 km. Find:

a) the annual depreciation for this particular year

b) its useful life if its scrap value is $12 000
Worked Example 21

A photocopier purchased for $10,800 depreciates at a rate of 20 cents for every 100 copies made. In its first year of use 500,000 copies were made and in its second year, 550,000. Find:

a) the depreciation each year

b) the future value at the end of the second year.

Unit cost depreciation equation

A future value after \( n \) outputs using unit cost depreciation can be expressed as:

\[
V_n = V_0 - nd
\]

where \( V_n \) is the value of the asset after \( n \) outputs and \( d \) is the depreciation per output.

If we were to use this equation with worked example 21

The rate \( d \) is \( \frac{0.20}{100} \) (20 cents per 100 copies)

The number of copies \( n = 500,000 + 550,000 = 1,050,000 \)

And \( V_0 = 10,800 \)

\[
V_n = 10800 - 1,050,000 \times \frac{0.20}{100}
\]

\[
V_n = 8,700
\]
**Worked Example 22**

The initial cost of a vehicle was $27,850 and its scrap value is $5,050. If the vehicle needs to be replaced after travelling 80,000 km (useful life):

a) find the depreciation rate (depreciation ($ per km))

b) find the amount of depreciation in a year when 16,497 km were travelled

c) set up an equation to determine the value of the car after travelling \( n \) km

d) find the future value after it has been used for a total of 60,000 km

e) set up a schedule table listing future value for every 20,000 km.

<table>
<thead>
<tr>
<th>Use, ( n ) (km)</th>
<th>Future value ( V_n ) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>20,000</td>
<td></td>
</tr>
<tr>
<td>40,000</td>
<td></td>
</tr>
<tr>
<td>60,000</td>
<td></td>
</tr>
<tr>
<td>80,000</td>
<td></td>
</tr>
</tbody>
</table>