Further Mathematics 2018

Core: Recursion and Financial modelling

Recurrence Relations, Interest and Depreciation

**Key knowledge**

- the concept of a first-order linear recurrence relation and its use in generating the terms in a sequence
- the use of first-order linear recurrence relations to model flat rate and unit cost and reducing balance depreciation of an asset over time, including the rule for the future value of the asset after \( n \) depreciation periods
- the concepts of financial mathematics including simple and compound interest, and depreciation

**Key skills**

- use a given first-order linear recurrence relation to generate the terms of a sequence
- model and analyse growth and decay in financial contexts using a first-order linear recurrence relation of the form \( u_0 = a, u_{n+1} = bu_n + c \)
- demonstrate the use of a recurrence relation to determine the depreciating value of an asset or the future value of an investment or a loan after \( n \) time periods, including from first principles for \( n \leq 5 \)
- use a rule for the future value of a compound interest investment or loan, or a depreciating asset, to solve practical problems

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8A Sequences

A list of numbers, written down in succession, is called a sequence. Each of the numbers is called a term. We write the terms of a sequence as a list, separated by commas. If a sequence continues indefinitely, or if there are too many terms to write them all, we use an ellipsis, ‘...', at the end of a few terms of the sequence like this:

\[12, 22, 5, 6, 16, 43, ...\]

The terms in this sequence of numbers could be the ages of the people boarding a plane. The age of these people is random so this sequence of numbers is called a random sequence. There is no pattern or rule that allows the next number in the sequence to be predicted. Some sequences of numbers do display a pattern. For example, this sequence:

\[1, 3, 5, 7, 9, \ldots\]

has a definite pattern and so this sequence is said to be rule-based. This sequence of numbers has a starting value. We add 2 to this number to generate the term 3. Then, add 2 again to generate the term 5, and so on. The rule is ‘add 2 to each term’.

![Sequence Example](image)

**Example 2** Generating a sequence of terms

Write down the first five terms of the sequence with a starting value of 10 and the rule ‘double the number and then subtract 3’.

**Solution**

1. Write down the starting value. 5
2. Apply the rule (double, then subtract 3) to generate the next term. \(2 \times 5 - 3 = 7\)
3. Calculate three more terms. \(2 \times 7 - 3 = 11\) \(2 \times 11 - 3 = 19\) \(2 \times 19 - 3 = 35\)
4. Write your answer. The sequence is 5, 7, 11, 19, 35, \ldots

Using a calculator to generate a sequence of numbers from a rule

All of the calculations to generate sequences from a rule are repetitive. The same calculations are performed over and over again – this is called recursion. A calculator can perform recursive calculations very easily, because it automatically stores the answer to the last calculation it performed, as well as the method of calculation.
Example 3  Generating a sequence of numbers with a calculator

Use a calculator to generate the first five terms of the sequence with a starting value of 5 and the rule ‘double and then subtract 3’.

Solution

Start with a blank calculator page

Press

- 2 on Home
- 1 New document
- 1 add calculator
- Type 5 5
- press enter

Next

- type \( \times \) 2 - 3

Press enter

Note: when you press enter, the CAS converts ANS to the value of the previous answer (in this case 5)

Pressing enter repeatedly applies the rule “x2-3” to the last calculated value, in the process generating successive terms of the sequence as shown.
8B Recurrence relations

A recurrence relation is a mathematical rule that we can use to generate a sequence. It has two parts:

1. a **starting point**: the value of one of the terms in the sequence
2. a **rule** that can be used to generate successive terms in the sequence.

For example, in words, a recursion rule that can be used to generate the sequence: 10, 15, 20,... can be written as follows:

1. Start with 10.
2. To obtain the next term, add 5 to the current term and repeat the process.

A more compact way of communicating this information is to translate this rule into symbolic form. We do this by defining a subscripted variable. Here we will use the variable $V_n$, but the $V$ can be replaced by any letter of the alphabet.

Let $V_n$ be the term in the sequence after $n$ iterations.

Using this definition, we now proceed to translate our rule written in words into a mathematical rule.

<table>
<thead>
<tr>
<th>Starting value $(n=0)$</th>
<th>Rule for generating the next term</th>
<th>Recurrence relation (two parts: starting value plus rule)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0=10$</td>
<td>$V_{n+1}=V_n+5$</td>
<td>$V_0=10$ $V_{n+1}=V_n+5$</td>
</tr>
<tr>
<td>$\text{Next term} = \text{current term} + 5$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:** Because of the way we defined $V_n$, the starting value of $n$ is 0. At the start there have been no applications of the rule. This is the most appropriate starting point for financial modelling.

**Worked Example 4**
Write down the first five terms of the sequence defined by the recurrence relation $V_0=9$, $V_{n+1}=V_n-4$ showing the values for the first 4 iterations.

**Example 5 Using a calculator to generate sequences from recurrence relations**

A sequence is generated by the recurrence relation $V_0 = 300$, $V_{n+1} = 0.5V_n - 9$.

Use your calculator to generate this sequence and determine how many terms of the sequence are positive.

**Solution**
Start with a blank calculator page
Press
- Type 300 3 0 0
- press enter

Next
- Type $\times 0.5 - 9$
- press enter

Continue to press until the first negative term appears and write your answer:
- The first 5 terms of the sequence are positive.

*Each time we apply the rule it is called an iteration.*
Modelling linear growth and decay

Linear growth and decay occurs when a quantity increases or decreases by the same amount at regular intervals.

An example of linear growth is the investment of money, such as putting it in a savings account where the sum increases over time.

An example of linear decay is the money owned to repay a loan, the sum of money owned will decrease over time.

**A recurrence model for linear growth and decay**

The recurrence relations

\[ P_0 = 20, \quad P_{n+1} = P_n + 2 \]

\[ Q_0 = 20, \quad Q_{n+1} = Q_n - 2 \]

both have rules that generate sequences with linear patterns, as can be seen from the table below. The first generates a sequence whose successive terms have a linear pattern of growth, and the second a linear pattern of decay.

<table>
<thead>
<tr>
<th>Recurrence relation</th>
<th>Rule</th>
<th>Sequence</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_0 = 20, P_{n+1} = P_n + 2 )</td>
<td>'add 2'</td>
<td>20, 22, 24, ...</td>
<td></td>
</tr>
<tr>
<td>( Q_0 = 20, Q_{n+1} = Q_n - 2 )</td>
<td>'subtract 2'</td>
<td>20, 18, 16, ...</td>
<td></td>
</tr>
</tbody>
</table>

Generally, if \( d \) is a constant, a recurrence relation rule of the form:

\[ V_{n+1} = V_n + d \]

can be used to model **linear growth**.

\[ V_{n+1} = V_n - d \]

can be used to model **linear decay**.

**Simple interest loans and investments**

If you deposit money into a bank account, the bank is effectively borrowing money from you. They will pay a fee to borrow your money which is called interest. If a fixed amount is paid into the account at regular time periods, it is called a simple interest investment.

If you borrow money from the bank and are charged a fixed amount of interest after regular time periods it is called a simple interest loan.

Simple interest is a special case of linear growth in which the starting value is the amount borrowed or invested is called the principal. The amount at each step is the interest and is usually a percentage of this principal.
Recurrence model for simple interest

Let $V_n$ be the value of the loan or investment after $n$ years.

Let $r$ be the percentage interest rate.

The recurrence relation for the value of the loan or investment after $n$ years is

$$V_0 = \text{principal, } V_{n+1} = V_n + D$$

where $D = \frac{r}{100} \times V_0$.

Example 6
Cheryl invests $5000 in an investment account the pays 4.8% per annum simple interest. Model this simple investment using a recurrence relation in the form:

$$V_0 = \text{the principal, } V_{n+1} = V_n + D \quad \text{where } D = \frac{r}{100} \times V_0$$

Example 7
Cheryl’s simple investment is modelled by $V_0 = 5000$, $V_{n+1} = V_n + 240$, where $V_n$ is the value of the investment after $n$ years.

a) Use the model to determine the value of Cheryl’s investment after 3 years.

Using CAS Calculator

Open up a new calculator page
Enter in the first value and press enter
Simply type +240 and press enter
Continue pressing enter and the value will continue to be added
b) When will Cheryl’s investment first exceed $6000 and what will its value be then?

Using CAS Calculator

<table>
<thead>
<tr>
<th>Year</th>
<th>Starting Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>$5000</td>
</tr>
<tr>
<td>2nd</td>
<td>$5240</td>
</tr>
<tr>
<td>3rd</td>
<td>$5480</td>
</tr>
<tr>
<td>4th</td>
<td>$5720</td>
</tr>
<tr>
<td>5th</td>
<td>$6200</td>
</tr>
</tbody>
</table>

It will be after 5 years with a value of $6200

**Depreciation**

Some items such as antiques, jewellery and real estate increase in value (appreciate or increase in capital gain. Computers, mobile phones, cars or machinery decrease in value (depreciate) with time due to wear and tear, advances in technology or lack of demand.

Depreciation is the estimated loss in value of assets. The estimated value of an item at a point in time is called its **future value** (book value).

When the value becomes zero, the item is **written off**. At the end of an item’s useful life its future value is called its **scrap value**.

**Future value = cost price – total depreciation to that time**

When book value = $0, then the item is said to be written off.

**Scrap value is the book value of an item at the end of its useful life.**

There are 3 methods in which to calculate depreciation:

1. flat rate depreciation
2. unit cost depreciation
3. reducing balance depreciation

Flat rate depreciation and unit cost depreciation can both be modelled using a linear decay recurrence relation.
**Flat rate (straight line) depreciation**

If an item depreciated by the flat rate method, then the value decreases by a fixed amount each time interval. It may be expressed in dollars or as a percentage of the original cost price.

As the depreciation value is the same for each interval, it is an example of straight line decay. This relationship can be expressed in the following recurrence relation:

<table>
<thead>
<tr>
<th>Recurrence model for flat-rate depreciation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $V_n$ be the value of the asset after $n$ years.</td>
</tr>
<tr>
<td>Let $r$ be the percentage depreciation rate.</td>
</tr>
<tr>
<td>The recurrence relation for the value of the asset after $n$ years is</td>
</tr>
<tr>
<td>$V_0 = \text{initial value}$, $V_{n+1} = V_n - D$</td>
</tr>
<tr>
<td>where $D = \frac{r}{100} \times V_0$.</td>
</tr>
</tbody>
</table>

**Worked Example 8**

A new car was purchased for $24,000 in 2014. The car depreciates by 20% of its purchase price each year. Model the depreciating value of this car using a recurrence relation in the form:

\[
V_0 = \text{initial value}, \quad V_{n+1} = V_n - D \quad \text{where} \quad D = \frac{r}{100} \times V_0
\]

**Worked Example 9**

The flat rate depreciation of a car is modelled by $V_0 = 24000$, $V_{n+1} = V_n - 4800$, where $V_n$ is the value of the car after $n$ years.

- a. Use the model to determine the value of the car after 2 years.

- b. If the car was purchased in 2014, in what year will the car’s value depreciate to zero?

**Using CAS Calculator**

- Open up a new calculator page
- Enter in the first value and press enter
- Simply type -4800 and press enter
Continue to press enter until the value is at, or below, zero

Figure out which year that value coincides with

<table>
<thead>
<tr>
<th>Year</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st year</td>
<td>24000</td>
</tr>
<tr>
<td>2nd year</td>
<td>19200</td>
</tr>
<tr>
<td>3rd year</td>
<td>14400</td>
</tr>
<tr>
<td>4th year</td>
<td>9600</td>
</tr>
<tr>
<td>5th year</td>
<td>4800</td>
</tr>
</tbody>
</table>

It will be after 5 years. If the first value is in 2014 + 5 years = 2019

### Unit-cost depreciation

Some items lose value because of how often they are used rather than how old they are. A photocopier that is 2 years old but has never been used could still be considered to be in 'brand new' condition and therefore worth the same, or close to, what it was worth 2 years ago. But if that photocopier was 2 years old and had printed many thousands of papers, it would be worth much less than its original value.

Cars can also depreciate according to use rather than time. People often look at the kilometres travelled before buying it. Cars of the same age can be worth very different amounts depending on the number of kilometres they have travelled.

When the future value of an item is based upon use rather than age, we use a unit-cost depreciation method, which is modelled using a linear decay recurrence relation.

#### Recurrence model for unit-cost depreciation

Let $V_n$ be the value of the asset after $n$ units of use.

Let $D$ be the cost per unit of use.

The recurrence relation for the value of the asset after $n$ units of use is:

$$V_0 = \text{initial value of the asset}, \quad V_{n+1} = V_n - D$$

#### Worked Example 10

A professional gardener purchased a lawn mower for $270. The mower depreciates in value by $3.50 each time it is used.

a. Model the depreciating value of this mower using a recurrence relation of the form

$$V_0 = \text{initial value}, \quad V_{n+1} = V_n - D \quad \text{where} \quad D = \text{the depreciation in value per use}$$
b. Use the model to determine the value of the mower after it has been used three times.
c. How many times can the mower be used before its depreciated value is less than $250?

Using CAS Calculator

Open up a new calculator page
Enter in the first value and press enter
Simply type - 3.50 and press enter

<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st use</td>
<td>270</td>
</tr>
<tr>
<td>2nd use</td>
<td>266.5</td>
</tr>
<tr>
<td>3rd use</td>
<td>263</td>
</tr>
<tr>
<td>4th use</td>
<td>259.5</td>
</tr>
<tr>
<td>5th use</td>
<td>256.5</td>
</tr>
<tr>
<td>6th use</td>
<td>252.5</td>
</tr>
</tbody>
</table>

b. after the 3rd use the value of the lawn mower is **$259.50**
c. the value is below $250 after the 6th use.
8D Rules for the nth term in a sequence modelling linear growth or decay

While we can generate as many terms as we like in a sequence using linear growth or decay, it is possible to derive a rule for calculating any term in the sequence directly. This is most easily seen by working with a specific example.

If you invest $2000 in a simple interest investment paying 5% interest per annum, your investment will increase by the same amount, $100 (5% of $2000), each year. The recursion relation would be:

\[ V_0 = 2000, \quad V_{n+1} = V_n + 100 \]

<table>
<thead>
<tr>
<th>Recurrence relation</th>
<th>General rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_0 = 2000 )</td>
<td>( V_0 = 2000 )</td>
</tr>
<tr>
<td>( V_{n+1} = 2000 + 100 )</td>
<td>( V_1 = 2100 )</td>
</tr>
<tr>
<td>( V_{n+1} = 2100 + 100 )</td>
<td>( V_2 = 2200 )</td>
</tr>
<tr>
<td>( V_{n+1} = 2200 + 100 )</td>
<td>( V_3 = 2300 )</td>
</tr>
<tr>
<td>( V_{n+1} = 2300 + 100 )</td>
<td>( V_4 = 2400 )</td>
</tr>
<tr>
<td>( V_{n+1} = 2400 + 100 )</td>
<td>( V_5 = 2500 )</td>
</tr>
</tbody>
</table>

Following this we can generate the rule \( V_n = V_0 + n \times 100 \)

The general rule can be created for simple interest investments and loans, flat-rate depreciation and unit-cost depreciation. The general rule for calculating the nth term is:

- \( V_n = V_0 + nD \) for linear growth
- \( V_n = V_0 - nD \) for linear decay

**Simple interest investments and loans**

Let \( V_0 \) be the initial value of the simple interest investment or loan.
Let \( r \) be the annual interest rate.

The value of a simple interest investment or loan after \( n \) years is

\[ V_n = V_0 + nD \quad \text{where} \quad D = \frac{r}{100} V_0 \quad \text{or} \quad V_n = V_0 + n \times \frac{r}{100} \times V_0 \]

**Flat rate of depreciation**

Let \( V_0 \) be the initial value of the asset.
Let \( r \) be the flat rate of depreciation.

The value of the asset after \( n \) years is

\[ V_n = V_0 - nD \quad \text{where} \quad D = \frac{r}{100} V_0 \quad \text{or} \quad V_n = V_0 - n \times \frac{r}{100} \times V_0 \]

**Unit-cost depreciation**

Let \( V_0 \) be the initial value of the asset.
Let \( D \) be the cost per unit of use.

The value of the asset after \( n \) units of use is

\[ V_n = V_0 - nD \]
Worked Example 12
The following recurrence relation can be used to model a simple interest investment:

\[ V_0 = 3000, \quad V_{n+1} = V_n + 260 \]

Where \( V_n \) is the value of the investment after \( n \) years.

a. What is the principal of the investment? How much interest is added to the investment each year?

b. Write down the rule for the value of the investment after \( n \) years.

c. Use a rule to find the value of the investment after 15 years.

d. Use a rule to find when the value of the investment first exceeds $10,000.

Worked Example 13
Amie invests $3000 in a simple interest investment of paying interest at the rate of 6.5% per year. Use a rule to find the value of the investment after 10 years.
Worked example 14
The following recurrence relation can be used to model flat rate depreciation of a set of office furniture
\[ V_0 = 12,000, \quad V_{n+1} = V_n - 1200 \]
Where \( V_n \) is the value of the furniture after \( n \) years.

a. What is the initial value of the furniture? By how much does the furniture decrease in value each year?

b. Write down the rule for the value of the investment after \( n \) years.

c. Use a rule to find the value of the investment after 6 years.

d. How long does it take for the furniture’s value to decrease to zero?

e. A photocopier costs $6000 when new. Its value depreciates at a rate of 17.5%. What is its value after 4 years?

---

Worked Example 15
A hairdryer in a salon was purchased for $850. The value of the hairdryer depreciates by 25 cents for every hour it is in use. Let \( V_n \) be the value of the hairdryer after \( n \) hours of use.

a. Write down a rule to find the value of the hairdryer after 850 hours of use

b. On average the salon will use the hairdryer for 17 hours each week. How many weeks will it take the value of the hairdryer to halve?
Worked Example
Fast Word Printing Company bought a new printing press for $15 000 and chose to depreciate it by the flat rate method. The depreciation was 15% of the prime cost each year and its useful life was 5 years.

a) Find the annual depreciation.

b) Set up a recurrence relation to represent the depreciation

c) Create a depreciation schedule for the useful life of the press and use it to draw a graph of book value against time.

Worked Example on CAS calculator
On a lists & spreadsheet page
- Label column A “n” and column B “Vₙ”
- Enter 0 to 4 in the n column and the starting value 15000 (V₀) in cell b1.

In cell b2
- Enter the equation “=b1-2250”
  This equation is just Vₙ₊₁=Vₙ-2250 found in part (b)
- Note: the 2250 is the annual depreciation found in part (a)

Press enter, then
- fill down ([menu] [3] [3]) until n=5
  Vₙ=3750 when n=5. So, this is the scrap value

Add a Data & Statistics page
- Label the x-axis “n” and the y-axis “Vₙ”
**8E Modelling geometric growth and decay**

Like linear growth and decay, geometric growth and decay are seen in everyday life. Some examples include the payment of compound interest or the depreciation of the value of a new car by a constant percentage each year. This sort of depreciation is commonly called *reducing balance depreciation*.

**A recurrence model for geometric growth and decay**

Geometric growth or decay in a sequence occurs when quantities increase or decrease by the same percentage at regular intervals.

Consider the recurrence relations below:

- $V_0 = 1, \ V_{n+1} = 3V_n$
- $V_0 = 8, \ V_{n+1} = 0.5V_n$

Both have rules that generate a geometric pattern, which can be seen in the table below.

<table>
<thead>
<tr>
<th>Recurrence relation</th>
<th>Rule</th>
<th>Sequence</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0 = 1, \ V_{n+1} = 3V_n$</td>
<td>‘multiply by 3’</td>
<td>$1, 3, 9, \ldots$</td>
<td><img src="image1.png" alt="Graph" /></td>
</tr>
<tr>
<td>$V_0 = 8, \ V_{n+1} = 0.5V_n$</td>
<td>‘multiply by 0.5’</td>
<td>$8, 4, 2, \ldots$</td>
<td><img src="image2.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

As a general rule, if $R$ is a constant, a recurrence relation rule of the form:

- $V_{n+1} = RV_n$ for $R > 1$, can be used to model geometric growth (increase)
- $V_{n+1} = RV_n$ for $R < 1$, can be used to model geometric decay (decrease)

We can use this information to model and investigate compound interest loans and investments, and reducing-balance depreciation.
Compound interest investments and loans

Most interest calculations are not as straightforward as simple interest. The more usual form of interest is compound interest where any interest that is earned after one time period is added to the principal and then contributes to the earning of the interest of the next time period.

This means that the value of the investment grows by increasing amounts each time period, unlike simple interest, where it grows by exactly the same amount each time.

A recurrence model for compound interest investments and loans that compound yearly

Let $V_n$ be the value of the investment after $n$ years.
Let $r$ be the percentage interest per compound period.

The recurrence model for the value of the investment after $n$ compounding periods is:

$$V_0 = \text{principal}, \quad V_{n+1} = R V_n$$

where $R = 1 + \frac{r}{100}$.

Worked Example 16

The following recurrence relation can be used to model a compound interest investment of $2000 paying interest at the rate of 7.5% per annum.

$$V_0 = 2000, \quad V_{n+1} = 1.075V_n$$

In the recurrence relation $V_n$ is the value of the investment after $n$ years.

a. Use the recurrence relation to find the value of the investment after 1, 2 and 3 years

b. Determine when the value of the investment will be the first to exceed $2500

Using CAS Calculator

Open up a new calculator page
Enter in the first value and press enter
Simply type $\times 1.075$ and press enter

Continue to press enter to generate the terms of the sequence, until the value exceeds $2500$

b. after the 4th year the value exceeds $2500$

c. Write down the recurrence relation if $1500$ was invested at a compound interest rate of $6.0\%$ per annum.
Worked Example 17
Brian borrows $5000 from a bank. He will pay interest at the rate of 4.5% per annum. Let $V_n$ be the value of the loan after $n$ compounding periods.

a. Write down a recurrence relation to model the value of Brian’s loan if interest is **yearly**.

b. Write down a recurrence relation to model the value of Brian’s loan if interest is **quarterly**.

c. Write down a recurrence relation to model the value of Brian’s loan if interest is **monthly**.

Reducing-balance depreciation
Reducing-balance depreciation is where the value of an asset depreciates geometrically. This means, each year the value of the asset decreases by a percentage of its value at the beginning of each time period, not simply based on the initial value (which is the case for flat-rate depreciation).

**A recurrence model for reducing balance depreciation**
Let $V_n$ be the value of the asset after $n$ years.
Let $r$ be the annual percentage depreciation.
The recurrence model for the value of the investment after $n$ years is:

\[ V_0 = \text{initial value}, \; V_{n+1} = R V_n \text{ where } R = 1 - \frac{r}{100} \]
Worked example 18
The following recurrence relation can be used to model the value of office furniture with a purchase price of $6900, depreciating at a reducing-balance rate of 7% per annum.

\[ V_0 = 6900, \quad V_{n+1} = 0.935V_n \]

In the recurrence relation \( V_n \) is the value of the office furniture after \( n \) years.

a. Use the recurrence relation to find the value of the office furniture after 1, 2 and 3 years, correct to the nearest cent.

b. Determine when the value of the investment will be less than $5000.

Using CAS Calculator

Open up a new calculator page
Enter in the first value and press enter
Simply type \( \times 0.935 \) and press enter

Continue to press enter to generate the terms of the sequence, until the value exceeds $2500

b. after the 5th year the value drops below $5000 for the first time.

c. Write down the recurrence relation if the furniture was initially valued at $7500 and is depreciating at a reducing-balance of 8.4% per annum.
8F Rules for modelling geometric growth or decay

While we can generate as many terms as we like in a sequence using geometric growth or decay, it is possible to derive a rule for calculating any term in the sequence directly. This is most easily seen by working with a specific example.

If you invest $2000 in a simple interest investment paying 5% interest per annum, compounding yearly. If we let $V_n$ be the value of the investment after $n$ years, we can use the following recurrence relation to model this investment:

$$V_0 = 2000, \quad V_{n+1} = 1.05V_n$$

<table>
<thead>
<tr>
<th>Recurrence relation</th>
<th>General rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{0+1} = 1.05 \times 2000$</td>
<td>$V_1 = 2100$</td>
</tr>
<tr>
<td>$V_{1+1} = 1.05 \times 2100$</td>
<td>$V_2 = 2205$</td>
</tr>
<tr>
<td>$V_{2+1} = 1.05 \times 2205$</td>
<td>$V_3 = 2315.25$</td>
</tr>
<tr>
<td>$V_{3+1} = 1.05 \times 2315.25$</td>
<td>$V_4 = 2431.01$</td>
</tr>
<tr>
<td>$V_{4+1} = 1.05 \times 2431.01$</td>
<td>$V_5 = 2552.56$</td>
</tr>
</tbody>
</table>

Following this pattern, after $n$ year’s interest has been added, we can write $V_n = 1.05^n \times V_0$

A rule for individual terms of a geometric growth and decay sequence

For a geometric growth or decay recurrence relation

$$V_0 = \text{starting value}, \quad V_{n+1} = RV_n$$

the value of the $n$th term of the sequences is generated by the rule:

$$V_n = R^n \times V_0$$

Exactly the same rule will work for growth and decay because it depends on the value of $R$, not the format of the calculation. This general rule can also be applied to compound interest loans and investment and reducing-balance depreciation.

**Compound interest loans and investments**

Let $V_0$ be the amount borrowed or invested (principal).

Let $r$ be the interest rate per compounding period.

The value of a compound interest loan or investment after $n$ compounding periods, $V_n$, is given by the rule

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

**Reducing-balance depreciation**

Let $V_0$ be the purchase price of the asset.

Let $r$ be the annual percentage rate of depreciation.

The value of an asset after $n$ years, $V_n$ is given by the rule

$$V_n = \left(1 - \frac{r}{100}\right)^n \times V_0$$
Worked Example 19
A principal value of $10000 is invested in an account earning compound interest at the rate of 9% per annum. The rule for the value of the investment after \( n \) years, \( V_n \), is shown below.
\[
V_n = 1.09^n \times 10000
\]

a. Find the value of the investment after 4 years, correct to the nearest cent.

b. Find the amount of interest earned after 4 years, correct to the nearest cent.

c. Find the amount of interest earned in the 4\textsuperscript{th} year, correct to the nearest cent.

d. If the interest compounds monthly instead of yearly, write down a rule for the value of the investment after \( n \) months.

e. Use this rule to find the value of the investment after 4 years (48 months)
Worked Example: Comparing Flat rate and reducing balance depreciation
A transport business has bought a new bus for $60 000. The business has the choice of depreciating the bus by a flat rate of 20% of the cost price each year or by 30% of the previous value each year.

a) Generate depreciation schedules using both methods for a life of 5 years.

<table>
<thead>
<tr>
<th>Flat rate</th>
<th>Reducing balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time $n$ (years)</td>
<td>Depreciation $d$ ($)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

b) Draw graphs of future value against time for both methods on the same set of axes.

c) After how many years does the reducing balance future value become greater than the flat rate future value?
Worked Example 20
A computer system costs $9500 to buy, and decreases in value with the reducing balance depreciation of 20% each year. A recurrence relation that can be used to model the value of the computer system after $n$ years, $v_n$, is shown below.

$$v_0 = 9500, \ v_{n+1} = 0.8v_n$$

a. Write down the rule for the value of the computer system after $n$ years.

b. Use the rule to find the value of the computer system after 8 years. Write your answer correct to the nearest cent.

c. Calculate the total depreciation of the computer after 8 years.

Worked Example 21
How many years will it take an investment of $2000, paying compound interest at 6% per annum, to grow above $3000? Write you answer correct to the nearest year.

Using CAS Calculator
After substituting all known values into the formula, use the ‘solve’ function on the calculator to find the value of $n$

Type solve into the calculator and place equation into the calculator leaving ‘n’ for the year

Once the formula is in type ‘,n’ before closing the brackets

$n=6.95851563316$ therefore to the nearest year is 7 years to grow above $3000$
Worked example 22
An industrial weaving company purchased a new loom at a cost of $56000. It has an estimated value of $15000 after 20 years of operation. If the value of the loom is depreciated using a reducing-balance method, what is the annual rate of depreciation? Write your answer correct to one decimal place.

Using CAS Calculator
After substituting all known values into the formula, use the ‘solve’ function on the calculator to find the value of \( r \) (not to be confused with \( R \))

Type solve into the calculator and place equation into the calculator leaving ‘\( r \)’ for the year
Once the formula is in type ‘\( r \)’ before closing the brackets

\[
r = 12.3422\ldots \text{ or } r=187.6577,
\]
In this case the answer of 187% would not make logical sense so the correct answer is 12.3% to 1d.p.

*note: If both make sense, choose the smaller of the two.

Worked Example
Suppose the new $15 000 printing press considered in Worked example 13 was depreciated by the reducing balance method at a rate of 20% p.a. of the previous value.
Generate a depreciation schedule using a recurrence relation for the first 5 years of work for the press and graph the future values against time.

Worked Example on CAS calculator

On a lists & spreadsheet page

- Label column A “n” and column B “\( V_n \)”
- Enter 0 to 5 in the n column and the starting value 15000 (\( V_0 \)) in cell b1.

In cell b2

- Enter the equation “=0.8×b1”

Note: This equation is just \( V_{n+1}=R×V_n \)
where \( R=0.8, \left(R = 1 - \frac{r}{100}\right), \text{and } r = 20\% \text{ p.a.} \)

Press enter, then

- fill down (menu3]3) until \( n=5 \)
\( V_n=4915.20 \) when \( n=5 \). So, this is the value of the press after 5 years

Add a Data & Statistics page

- Label the x-axis “n” and the y-axis “\( V_n \)”
8G Nominal and effective interest rates

Nominal interest rate

Compound interest rates are usually quoted as annual rates, which is referred to as the **nominal interest rate** for the investment or loan. Sometimes an annual rate is quoted, but the interest is actually calculated according to a different time period, such as monthly or quarterly.

It must be assumed that there are:
- 12 equal months in every year (even though some have more days than others)
- 4 quarters in every year (a quarter is 3 months)
- 26 fortnights in a year (even though there are slightly more)
- 52 weeks in a year (even though there are slightly more)
- 365 days in a year (ignore the existence of leap years)

A nominal interest rate is converted to a compounding interest rate by dividing by these numbers.

**Worked Example 23**

An investment account will pay interest at the rate of 3.6% per annum. Convert this to the following:

a. A monthly rate

b. A fortnightly rate

c. A quarterly rate
Effective interest rate

As a general principle with the compound interest, the more frequently interest is calculated and added to your investment or loan (the compounding period), the more rapidly the value of your investment or loan increases. This is illustrated in the table below which compares the value of a $5000 investment paying a nominal interest rate of 4.8% per annum with the value of the investment if the interest is calculated on a quarterly or monthly basis, rather than just yearly.

<table>
<thead>
<tr>
<th>Principal of investment: $5000</th>
<th>Nominal annual interest rate: 4.8%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of investment for interest earned at the rate of:</td>
<td>4.8% per annum</td>
</tr>
<tr>
<td><strong>Month</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td>0</td>
<td>5000.00</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
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<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

It is evident that the more often the interest is compounded, the greater the amount of interest that is earned over time. This means that an investment where the interest is calculated more regularly has a higher effective interest rate than one which is calculated at a longer time intervals. Instead of completing a whole table, the effective interest rate can be calculated using the formula below:

**Effective interest rate**

The effective interest rate of a loan or investment is the interest earned after one year expressed as a percentage of the amount borrowed or invested.

Let:
- \( r \) be the nominal interest rate per annum
- \( r_{\text{effective}} \) be the effective annual interest rate
- \( n \) be the number of times the interest compounds each year.

The effective annual interest rate is given by: 

\[
r_{\text{effective}} = \left(1 + \frac{r}{100} \right)^n - 1 \times 100\%
\]
Worked Example 24
Brooke would like to borrow $20000. She is deciding between two loan options:
- Option A: 5.95% per annum compounding weekly
- Option B: 6% per annum compounding quarterly

a. Calculate the effective interest rate for each investment.

b. Which investment option is the best and why?

**Remember**
For any compound interest loan or investment, increasing the number of compounds per year will increase the total interest earned or paid.

Worked Example 25
Marissa has $10,000 to invest. She chooses an account that will earn compounding interest at the rate of 4.5% per annum, compounding monthly.

Use a CAS calculator to find the effective rate for this investment, correct to three decimal places.

<table>
<thead>
<tr>
<th>Using CAS Calculator</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type into the calculator</strong> ‘eff’ <strong>and open brackets</strong></td>
</tr>
</tbody>
</table>

| **The first number entered into the brackets needs to be the annual interest rate, in this case 4.5** |
| **Then and comma and then the number of compounding periods per year, in this case monthly = 12** |

| **The effective interest rate correct to three decimal places is 4.594%** |